Who Carved Up The Integers?

(They Never Died)

or

A Monograph on the Importance of the Integers and the Possibility of a Historically and Philosophically Self-Conscious Mathematics

or

The Method of Exhaustion, the Endless Screw and the Linear Plot

Nicola Graves Gregory

Contents

Acknowledgements

Foreword or Apologia

This is an unfinished work. It is the text
of what was to be the first chapter of a
Ph.D., abandoned some years ago. It contains
some ideas which I and others, more
academically authoritative than me, think
could be useful in the history and philosophy
of mathematics. I intend to delve further
into some of the directions outlined here,
but not all. I am publishing the text as it
stands in the hope that some other people may
take up and investigate further some of the
possibilities suggested. I would be very
pleased to get feedback from anyone
interested.

I apologise for the incompleteness of the
references. To recover them would take more
time than my familial responsibilities and my
need to research further allow me.

With respect to the content of the text, I
consider the most important ideas to be in
section 4. Section 3 is included because it

sets the scene for section 4. Sections 1 and 2 are included because some mathematical historians who have read this work find them interesting. To be really up-to-date, these latter two sections should be revised in the light of Philip Kitcher's **The Nature of Mathematical Knowledge** which appeared shortly after I had given up working for a Ph.D., but, as I have explained, I have neither the time nor motivation now to embark on such a task.

1. Introduction

I originally had two main motivations in my
research. One was a concern with the
phenomenon of the integers. I intended to
examine how and why attempts to investigate
the qualities of the integers gradually lost
importance after the discovery of the
existence of irrationals, and to examine what
meaning the essential concern of the
Pythagoreans with the integers might have for
us today. This led on to considerations of
the changes in the concept of number.

The other motivation was a certain
dissatisfaction with the predominant trends
in current history and philosophy of
mathematics, in that they fail to adopt a
critical perspective on present-day
mathematics and are thus unable to play an
active part in determining the telos of
mathematical development, i.e. to offer con-
structive suggestions as to possible and
desirable directions for mathematics. I felt
that a form of synthesis, historical
philosophy or philosophical history (as
suggested by Lakatos and exemplified in his
Proofs and Refutations) might offer a real
possibility of such a perspective. Certainly
contemporary mathematical histories are not
philosophically oriented in a presently
active sense, and contemporary mathematical
philosophies define themselves ahistor-
ically.[1]

In this investigation, whilst beginning to examine these two initially distinct topics, I hope to indicate that they are, in fact, linked; that the attitudes and implicit assumptions which underlie current historical and philosophical perspectives are rooted in those that have governed the development of our conceptualisation of number and the perception of the integers.

Before going into more detail in justifying and amplifying the above statements and examining their implications, I would like to outline an argument for a historical philosophical perspective. (The irony of the form of my argument will shortly be apparent.)

1)   Mathematical concepts and constructs are historically situated.[2]

        This statement does not conflict with a Platonist view of mathematics; it concerns mathematical means, not its telos.

        To deny this statement would imply the belief that there had been no real change in mathematics (only a change in language), not a view generally maintained by mathematicians or metamathematicians.

2)   We have choice as to how mathematics is to develop.

        The contrary belief implies some kind of determinism.

Either a) behaviourist, i.e. we are out of our own control.

or   b) 'mystical' platonist, i.e. mathematical truths are absolute and eternal; mathematicians do not choose what they do mathematically, they can only try to follow their intuition as a guide to the truth.[3]

If we accept 1 and 2, i.e. do not accept 2a or 2b, then we are presented with the question:

3) In what directions do we think it desirable and possible for mathematics to develop?

This contains two questions:

3a) What are our criteria for deciding in what directions we wish mathematics to develop?

What alternative aims, modes, methods are possible for mathematics?

This problematic is ignored in contemporary mathematical philosophy, in that none of the mathematical schools concern themselves directly with the development of mathematics. For all their differences, the main debate and praxis is concerned with what constitutes mathematical rigour, almost the opposite

approach; given a body of mathematical knowledge, what can be called certain?

Since this problematic has not been explicitly, directly confronted as such before, we are not clear about the directions and aims implied in present mathematical praxis. So we have a preliminary question:

4)   How can we clarify the underlying intentional[4] meaning structure, contained in contemporary mathematics?

Question 3a obviously involves many questions including that of the social role of mathematics, one which has mostly been avoided in the present day, although it was considered in the Greek era.[5] I shall not attempt to deal with it here.

Questions 3b and 4 present an extremely difficult task, since, as Husserl points out, our language, understanding and perception are permeated with the 'sedimentation' of history.[6] To begin to extricate ourselves from this mesh, we must re-examine the history which created it - both the steps which led to the present mathematical configuration, and the alternatives that were rejected. Although it is not possible to know definitively what was entailed in past possibilities, it is sufficient if we can begin

A)   to appreciate the nature of the contingent element in alternative

mathematical projects,

B)   to understand why, in a given hist-
        orical context, alternative projects
        were not adopted,[7]

and hence

C)   to understand the implications of
        the projects that were favoured.

Any significant progress here might enable us
to tackle a perhaps more controversial
project:

D)   to re-evaluate whether some
        previously rejected projects might
        have some meaningful content for
        mathematics now.

For examples of such projects and
controversies I shall largely draw on Greek
and Renaissance/Reformation periods, which I
see as crucial in determining the content and
modes of our present mathematics. In par-
ticular, I regard the decline of Pythagorean
number theory, in status and (as I shall
argue) in living content, as one of the
earliest examples of choice in the history of
mathematics, and extremely valuable to
examine.

First, I shall look at the current state of
the history of mathematics, in support of my
contention that its lack of critical
perspective produces serious weaknesses. It
will be necessary to alternate between
philosophical-historical concerns and those
of number for a while: at times it may seem
as though the windings and turnings of my
argument have lost sight of their goal. But
in fact I am throughout concerned with what I
see as the contemporary impasse of
mathematical history and philosophy: and
further concerned to show that this impasse,
when traced back to its historical origins,
turns out to be deeply related to the
integers.

2. Mathematical History

Most research into the history of mathematics
has attempted to understand the past from the
perspective of contemporary formulations and
concepts. This approach is valuable   in
picking out threads of continuity; but it
also   leads   to   misinterpretations   and
distortions of the past. A   misleading
emphasis is given to conceptual developments
which can be seen as steps leading to present
formulations, whilst those perspectives which
in fact do not flow into the present are
treated,   in   effect,   either   as   pre-
mathematical   or   non-mathematical,   as
'anticipations' or 'false   trails',   thus
obscuring not only the internal phenomenology
of the past, but also the real nature of
mathematical development.[1]

Such a Whig theory of mathematical history -
one which regards present concepts as the
logically inevitable apex of mathematical
investigation - is basically ahistorical: it
makes it hard or impossible to see current
mathematics,   like   all   its   earlier
equivalents,   as a moment in an ongoing
historical process,   where by no   means
inevitable choices are constantly being made
between alternative projects and paradigms.[2]
It is for contingent, not absolute, reasons
that various historical projects have gone to
the wall; and there are   certainly senses in
which the array of problems tackled   by
mathematics is historically arbitrary.[3]

Because historical premises are not generally
explicated, the underlying premises are not

readily seen. When one does articulate the
approach which most historians of mathematics
have adopted, it becomes apparent that, in
fact, a model of deductive logic (an element
of contemporary mathematical theory) is
imposed upon history on two distinct levels:
on one level, clearly, in the Whig
perspective outlined, which interprets
earlier views and concepts as 'goodies' or
'baddies', true or false, right or wrong,
according to whether they are perceived as
being close to, or far from current views and
concepts; on another level (where the
implicit extension of an intramathematical
attitude is perhaps not so immediately
obvious), in the perspective of historians -
sometimes consciously attempting to avoid the
Whig distortions - who seek the _true_ history,
the _real_ reasons for events etc., and fail to
come to terms with the historical
situatedness of their own perspective.

Certainly, in both these cases the picture is
shaded, not sharp black-and-white, a
multivalent truth function rather than a
bivalent one. But the fundamental point is
that a measure (albeit fuzzy) is imposed upon
history: in the first case it is actively
imposed upon historical phenomena; in the
second, it plays a more passive role with
respect to historical interpretations.

The historical roots of the black/white view
could be said to lie in Parmenides'
philosophy. If we do not accept the
Parmenidean conclusion, but rather believe
that change is real, then we see that, as
D.L.Miller[4] says, 'the emergent gives rise to

a new perspective, a new past' - what seems
plausible or important in a historical
explanation changes as our mental frameworks
change. This does not mean that history must
necessarily act as a passive support for the
status quo; it can provide a source of
alternative perspectives, revealing hidden
assumptions in those currently held. On
examining the paths that led to our
conceptualisations we may find places where
the ideas that prevailed, did so not by
transcending the former contradictions, but
for contingent reasons, and re-examination of
these issues may lead to new resolutions,
other possible directions.

The 'sedimentation' of history permeates our
attitudes and understandings to such an
extent that reality is now seen as having an
exact mathematical nature; it has been
forgotten that this was a superstructure
imposed as a hypothesis, because the
hypothesis has proven so successful.[5]
Certainly the technical usefulness of our
current mode of mathematics can not be
denied. It has provided a wide variety of
models that can accommodate quantitative
change (one of the major advances from the
Greek stage) and so serve for predictive
scientific theories; but this was the
original intention. Although it may be argued
that our pure mathematics does not share this
goal explicitly, nevertheless it has
developed within conceptualisations defined
by scientific terms, and even, at a distance,
by technological concerns - the demand for
the calculus which arose from ballistics
problems is just one example[6]. Obviously we

have to examine more closely the areas and
ways in which, for instance, the
philosophical requirements of the Greeks are
embedded in mathematical concepts and
attitudes, as well as the metaphor
transference[7] (e.g. from the predominantly
mechanical subject matter of the Renaissance)
mediated through the seemingly neutral,
abstract mathematical function. But we must
recognise that the success it has achieved in
its chosen direction in no way validates its
being the only direction possible for
advance. We can draw an analogy with someone
who wishes to leave a town: he is free to go
in any direction; having journeyed, he may,
at any time, measure his progress in terms of
his distance from the starting point, but
this offers no means for qualitative
comparison between the place that She has
reached and the other alternatives.

There is the possibility of such a
qualitative evaluation if we try to recover
the mathematical problems of the past as they
were formulated and understood in their own
context, examining not only the ideas that
have survived into the present, but also
those that have been abandoned. We can then
begin to understand the dynamics of
mathematical development and to obtain a more
critical perspective on contemporary
mathematics. By attempting to appreciate the
meanings and implications of concepts and
attitudes which persisted into the present
mathematical corpus, we begin to have a
context in which to discern its underlying
intentional meaning structures as well as the

possibility of examining the implications which alternative directions might have for us now.

I see this task of extricating ourselves, distancing ourselves from our ideological context as an orientation; I do not consider that the phenomemological 'epoche' admits of an absolute consummation, since that, like the historical perspectives already discussed, would imply transcending our historical situation. This is the goal that Klein and Husserl in fact set themselves. For Klein the final task arising from the attempt to reactivate the 'sedimented history' of the 'exact' nature is 'the rediscovery of the prescientific world and its true origins'.[8] The situation envisaged is, like that of the Cartesian doubt, not a real doubt, but a pretence; we can not actually put ourselves into the pre-knowledge situation. The attempt to come closer to understanding the motivations and meanings (implicit and explicit) contained in earlier mathematical decisions must be seen in the context of an attempt to understand our own position.

Husserl was concerned to uncover the 'essential' history of mathematics and thus the essential nature of mathematics. For him, to understand something is a positive act, a living moment; so one of the basic questions which he asks about mathematical development is: how could it be possible to relive all the moments of understanding that are necessary in a mathematical proof in order to progress to further knowledge? Since he sees

knowledge as real, certain only in the moment
of its realisation ('Verwirklichung'), and it
is impossible simultaneously to realise all
the elements in a proof, (which would entail
realisation of the complete reductio), so he
creates another category of thought to cover
this case in mathematics, namely thought
which has the potentiality of being
reactivated.[9] The use of reactivatable
elements in a proof would guarantee the
soundness (reactivatability)[10] of the whole,
so that a vast edifice of mathematical
knowledge could be built up from certain
basic elements, provided that each step in
the construction satisfied the criterion of
reactivatability.

He does not consider the question which is
prime for Lakatos (which has been put by
mathematicians through the ages), namely, how
does one arrive at mathematical concepts or
discover (or create) new theorems in the
first place - a stage which is necessarily
prior to any attempt at proof. Focussing on
mathematical development rather than
certainty, Lakatos advocated a different
attitude to the concept of proof from the
present norm, and from Husserl's idealisation
which is in fact structurally the same as the
norm but projected onto a deeper meaning
level. Rather than seeing the attempt to
prove a theorem as the attempt to establish
it beyond doubt, in which case the appearance
of counterexamples is regarded primarily as a
failure of the theorem (or sub-lemmas),
Lakatos was concerned not with an absolute
result, but with the process involved in
attempts to prove a theorem. By setting out

the various steps of the argument, by looking
for local and global counterexamples (i.e.
counterexamples to the sublemmas, and
counterexamples to the principal theorem),
the hidden assumptions in the concepts used
may be revealed, opening the way to
realisation of more general concepts
underlying those initially considered. Proof
attempts and counterexamples thereby act as
two poles in a continual process of improving
conjectures, refining our mathematical ideas.

In fact Husserl's initial analysis, without
the somewhat ad hoc creation of
'reactivatability', is a valid approach to a
phenomenology of mathematics which
complements Lakatos' suggested methodology.
It is because it is not possible to carry out
a simultaneous reactivation of all the steps
involved in a mathematical argument, that all
proofs are temporary and non-absolute. They
are necessarily partial, since the limits of
the definitions of concepts can not become
clear until they are seen to be contained in
deeper, more general concepts.[11]   Because it
is impossible to apprehend a proof in its
totality back to the basic premises, attempts
to reactivate certain elements of a proof, in
their context or in another wider context,
can reveal new ideas that were obscured by
implicit assumptions in the original proof.

By following the development of a geometrical
problem into topological concepts, Lakatos
showed that the process he recommended as a
methodology was that which actually took
place over the course of history - but
unconsciously (and in fits and starts) since,

in keeping with the prevalent attitude to
proof, counterexamples were regarded as
'monsters', causing mathematical reactions
which he characterised as 'monster-barring',
'monster-adjusting' etc., rather than
stimulating careful re-examination of the
steps of the proof to discover precisely
which steps or concepts were called into
question in each case.

Lakatos' suggestion was that this process of
improving conjectures should be adopted
consciously as a methodology of mathematical
development. In effect, he argues that
mathematics already has a methodology latent
within present forms; it is only the rigid,
static interpretation of these forms which
has prevented perception of the essential,
complementary dynamic that they contain. It
requires only a fluid rather than a static
attitude to proof in order to appropriate and
activate this methodology (to work with it
rather than against it).

Lakatos' critique focuses on attitudes to the
concept of proof which determine reactions to
the occurrence of counterexamples, thus
affecting the development of any given
problematic in mathematics. In a similar way
to Lakatos, I am concerned with underlying
attitudes which have influenced the
development of different modes of
mathematics.

The way in which most mathematical historians
regard as deviants earlier mathematical
projects which do not translate or conform to
the current, conventional mathematical ideas,

is similar to the view of counter examples
as monsters ; the insidious extrapolation of
the mathematical-logical view of
contradiction from static to temporal
phenomena manifests itself in the inability
to come to terms with real change.

Szabo suggests that the reductio ad absurdum
form of proof which replaced earlier more
illustrative forms, was adopted from
Parmenidean logic.[12] Whether or not this was
the case (his argumentation is convincing)
the two forms are effectively the same. This
shift in the mode of proof results in a move
away from the intuitive reasoning which led
to the conjecture initially.[13] This
particularly affects the proofs of number
theoretical theorems[14], which is surely a
factor in the subsequent growth of geometry
and the impoverishment and degeneration of
arithmetike. I shall argue that this was also
the beginning of the notion of mathematics as
being more essentially concerned with
quantity than qualities.

Obviously these questions require a much
fuller investigation. Here I only wish to
suggest that at the very inception of the
mathematico-logical method which was to prove
so powerful in the development of
mathematics, it brought about changes which
profoundly affected the telos of mathematics;
the implications of some of these changes
were only articulated much later, and it was
still later before some of the inherent
limitations began to be realised and the
question as to the limits of validity (and
usefulness) of the method recurs.[15]

Husserl and Lakatos are important, because
they open up debates in little touched areas
of mathematical philosophy: Husserl's
discussion relates to the ontology of
mathematical knowledge; Lakatos is concerned
with the methodology of mathematical
development. (The latter's study, using
historical research to offer a constructive
critique of current mathematics and
mathematical philosophy, also serves as a
relevant example of the results possible
through combining historical investigation
and philosphical analysis.) I think it can be
shown that Husserl and Lakatos, via different
paths, in fact enter one space, a
philosophical area peculiar to mathematics
with its unique relation to the life-world.

Now, I shall return to the more particular
focus of my inquiry: the integers and the
concept of number.

3. Number theory:
towards a selective history

In this section I shall be dealing mostly
with topics which arose out of my interest in
the integers; but not directly with the
integers themselves. As necessary
background, therefore, I shall insert here a
short eulogy to natural number.

I have long been intrigued by the dual nature
of the integers. In the first place they are
our archetype of a discrete well-ordering.
(It is this aspect which is taken as their
defining characteristic in attempts to
'found' the integers on various set theories,
i.e. to construct set theoretic models that
functionally approximate to the natural
numbers.) They appear as a monotonous
repetition of a single relationship ad
infinitum: $1<2<3<4.....$,very useful for
counting sheep, but scarcely exciting.

And yet as we look closer - as our range of
numerical operations expands - we perceive
more and more complex structures generated by
this seemingly banal series: more varied
relationships between the numbers, in terms
of which we begin to appreciate the integers
as individuals with different
characteristics. As the range of structures
that we are able to identify increases in
diversity and complexity; as, where we once
saw undifferentiated extension, we now see a
finer, more subtle web of interlaced and
distinct entities; so we see this web to have
been latent in our primary, deceptively
simple sequence. When an ever finer grain

emerges at every increase in our power of definition - the subtlety of the number series seeming always one step ahead of our subtlety - how can we avoid the induction of the image of the natural numbers as the eternal source and limit of our pattern-making?[1] With the primes, perhaps, as perpetual jokers, continually escaping the webs we weave.

Well, yes, I am in love with the natural numbers; and, as I hope will emerge in this essay, I see the real number line and the complex plane as natural extensions of the integers - extensions whose developments and properties seem at times, like the paths in Alice's looking-glass garden, to lead away from their source, but always finally return to it.

When we look at contemporary mathematical work on the integers, although various mathematicians have commented on the integers in this vein, very little actual mathematical work corresponds to this attitude.[2] There are two main areas of contemporary mathematical work on the integers: foundational studies and number theory.

In the former, the approach adopted to the integers is that it is necessary to 'found' them in some form of set theory.[3] This results in a model of the integers as a class, taking their most obvious attribute, the discrete well-ordering, as the defining characteristic, and (since the stress is on the homogeneous aspect of their nature) reveals nothing about the complex

interrelations of the integers (i.e. about
the integers as individuals), but then that
is not its aim.

Thus foundational work has a more clearly
articulated aim than contemporary number
theory: to ensure secure foundations for the
mathematical edifice. It is perhaps partly
for this reason that it enjoys a higher
status; even though the strong form of this
aim - that the system should be consistent
and complete, was shattered years ago by
Godel's theorem. Certainly foundational,
mathematico-logical concerns continue to
dominate mathematical philosophy. But I shall
argue that the driving force behind this
emphasis is an attitude which has haunted
mathematics and its meta-disciplines,
mathematical history and philosophy since
Parmenidean bivalent logic was incorporated
into mathematics as a guarantor of certanty.
Wittgenstein drew attention to it in **Remarks
on the Foundations of Mathematics,** where he
writes,

> My aim is to alter the attitude to
> contradiction and to consistency proofs.
> Not to shew that this proof shews some-
> thing unimportant. How <u>could</u> that be so?[4]

This attitude, which I am tempted to call
'contradiction-phobia' is a fundamental block
that I consider we are in a far better
position to overcome now than in the Greek
era.

Needless to say, I feel that my interest in the integers is closer to the other main approach to the integers, via number theory, which takes the integers as given and claims to concern itself with the integers as individuals, their characteristics and interrelations. For this reason I wished to examine how and why, since the time of the Pythagoreans, number theory gradually declined in status and, as I shall argue, lost sight of its original telos, thus losing touch with its living content, losing wholeness, coherence; and to examine whether there might be a way in which the essential concern of the Pythagoreans could be meaningful today.

For the Pythagoreans, number theory or arithmetike was the basis of their metaphysical science[5] - their monadology was an attempt to discover the relationships of the universe, which they originally believed could be described totally in terms of integers and ratios of integers. In their mathematical work they both extended the range of possible operations with numbers (in geometry, theory of ratios etc.) and simultaneously objectified the properties of the individual numbers which emerged[6]. That is to say, they were concerned to develop more complex structures which could accurately describe the complicated phenomena in the world, but not to lose sight of the role which the integers played in these developments. By naming the qualities, properties which are the complementary result of an operational pattern, an alternative mode of describing it, it may prove possible

to perceive relationships between these properties themselves - i.e. another mathematical structure - as, of course, has proved to be the case with the properties which the Pythagoreans abstracted; these relationships between the Pythagorean-derived properties are still the foundation of number theory today. It is possible gradually to build up a more complete picture of each individual integer in terms of its properties - in which case, in whatever context one is dealing with a particular integer, one's awareness of its various attributes may yield simultaneously a new understanding about the nature of the context (thus revealing new possibilities) and about the nature of the integer, for example another attribute, or a meta-relation of properties.

Of course, I am not claiming that the Pythagoreans consciously adopted this twofold approach. In fact, whereas the current mathematical approach emphasises operational structures, the Pythagorean emphasis seems originally to have been to discover qualities of the integers - structural developments serving as tools to this end. But the resultant of their work was a balance between these two aspects of development (the dependence of the hypostatisation on operational developments is not matched by the inverse necessity) until the discovery of the existence of irrationals.

There is a tendency to depreciate the Pythagoreans' concern with the qualities or forms ($\epsilon \H{\iota} \delta \eta$) of the integers as number mysticism, which I feel to be questionable.

There are two interwoven strands in their
approach: the attempt to associate different
numbers and ratios with human
characteristics, moral attributes etc.; the
other we could retrospectively
(anachronistically) call defining equivalence
classes of integers (largely from the forms
which arose out of the figurate
representations). The former does not
invalidate the latter, as the attempts of the
Merton scholastics to quantify such entities
as love etc. do not invalidate their work on
dynamics. I shall not now consider the
question of the possible importance of a
'mystical' or 'metaphysical' perspective in
furthering the more limited mathematical
praxis - a question which has recurred with
regards to the relation between Newton's
alchemical researches and his accepted
mathematical work. [7] I only wish to point out
that the usual connotation of 'mystical',
namely 'opposed to reason', does not seem to
apply to the Pythagoreans. Their transcendent
telos is rooted in a materialist perspective
- they are described by Aristotle, together
with the 'physiologists' as being of the
opinion that being extends no further than
sense perception[8]. Aristotle also stresses
that it is Plato who first makes number
separable from objects of sense, whereas for
the Pythagoreans the monads have magnitude[9].

The discovery that there were lengths that
could not be expressed as integers or ratios
of integers meant that the Pythagorean
monadology was no longer tenable as a global
philosophy. The original concept of number
was maintained. The term 'number'

('arithmos') was used only for the integers
greater than one; 'one' (the 'monas') was the
unit and unity, and was considered the
principle or beginning of number and as such
had a different status from the numbers which
it generated: 'two' was sometimes similarly
excluded from the realm of number, a
reasoning which stems from the Pythagorean
identification of 'one' and the odd numbers
with 'limited', opposing 'two' (or the
'dyad') and the even numbers as
'unlimited'[10], but this perception of 'two'
was not observed so strictly; fractions were
not considered to be numbers but solely as
ratios of integers, since the 'one' was
essentially indivisible.

Since the irrationals are not generated by
the 'one', nor do they reduce to ratios of
integers, they had no place in this system
(they did not even have a justified
operational framework until Eudoxus adapted
the theory of proportion to this purpose),
and they were incorporated into mathematics
as geometrical magnitudes. There was now a
rigid ontological distinction between the
objects of study of arithmetike and geometry:
'number' is discrete, a multitude of
indivisible units; 'magnitude' is continuous,
an infinitely divisible spatial measure.[11]

From this time on, the development of
geometry began to outstrip that of
arithmetike - before, with the figurate
representation of numbers and the theory of
ratios, geometry and arithmetike were closer
and shared developments; now their objects of
study were separate. It could be argued that,

in any case, number theory had already
exhausted the possibilities for deriving
terminology from geometric forms. This is a
separate question; here, what is relevant is
that number was excluded from the study of
geometry; when the distinction was bypassed
in the Renaissance, the content of number
theory was already determined and the living
relationship between the study of the
integers and other mathematical areas was not
renewed.

From this time also, there was a shift from
illustrative demonstration to more strictly
logical proofs - relying more on reductio ad
absurdum. Szabó shows that for number
theoretical theorems, such indirect proofs
were sometimes substituted unnecessarily (in
terms of rigour) and perversely (in terms of
intuitive value[12]). By the time of Euclid's
compilation, the appended diagrams served no
useful purpose, representing discrete numbers
by line segments.

Also in this period the study of number
itself was split into two disciplines -
arithmetike and logistike. The dividing line
was never unequivocally established, but one
fairly common factor in the various versions
given is that arithmetike deals with the
'eide' - forms, kinds, species - of number;
whereas logistike deals with the 'hyle' of
numbers - the quantity, the material, the
amount that they represent.[13] The verbal
roots of their mathematical meanings are
respectively, 'arithmein' - to count, and
'logizmein' - to calculate. Disregarding for
the time being the original underlying

reasons for this distinction, it is clear that it was uncritically accepted by the neoplatonists as a rigid separation (Plato's suggested refinement of a further classification into theoretical and practical areas of each was not followed). The attempts at clarifying the distinction consisted in trying out different formulations for it and adjusting the classifications of the existing mathematical subject classification accordingly, rather than questioning its basic premises. So the study of the natures of numbers and that of numerical operations were seen as separate rather than as complementary and mutually stimulating.

It is relevant that Diophantine analysis, which was vitally important for the growth of modern algebra and which, according to the Platonist distinctions, should have appeared as logistike, or theoretical logistic (when Vieta takes it up, he reverts to this term) in fact appeared in his **Arithmetica**, i.e. Diophantus disregarded this distinction. He also moved away from the mainstream in that he did not use the Euclidean proof form and he accepted fractions as numbers.

So it seems that we have extricated three important factors contributing to the decline in status and/or content of number theory in the Greek era:

i)   the rigid ontological distinction between number and magnitude,

ii)   the emphasis on logical proof (particularly bivalent logic) as

opposed to illustrative demon-
stration,

iii)   the hypostatisation of the
arithmetike/logistike distinction.

Obviously these points require further
examination, but now to continue our whistle-
stop history of number theory. In the middle
ages interest in the integers was primarily
in terms of numbers as religious or moral
symbols; this is certainly of interest in
some respects but it is not directly relevant
to our inquiry at present. From the
Renaissance until the 19th century number
theory basically consisted of a range of
seemingly rather disparate (albeit
stimulating and important) problems, such as
the question of the distribution of primes,
Fermat's last theorem etc., that appeared to
have arisen almost accidentally in the course
of its history. It had become an area of
mathematics that lacked an inner sense of
direction and wholeness, having been content
to assume problems which involved terms which
figured in the Greek number theory: its only
claim to wholeness was the tenuous continuity
with the Greek discipline, which it
maintained by preserving the superficial
content of their concern - one might say that
number theory had petrified.

In the 19th century it gained coherence when
Gauss (who considered number theory to be the
queen of mathematics) published his
Disquisitiones Arithmeticae which extended
the notion of integers to include complex
integers, laying the ground for algebraic

number theory. Analysis began to be used in number theoretical proofs, giving rise to analytic number theory. But the emphasis continues to be on other branches of mathematics acting as investigatory or proof machinery with respect to number theoretical problems whose roots go back to Pythagorean arithmetike. Although analytic and algebraic insights have extended and deepened the structural vocabulary of number theory, it has not appropriated the expanded field of operations involving number (the extensions of logistike) as a potential source for developing its basic descriptive terminology for the integers themselves.

The phenomenon of the occurrence of particular integers in diverse mathematical fields is not examined for the significance they may have in terms of a nature or characteristic of the integer involved; even though, particularly in algebraic geometry and topology, such phenomena are increasing and it is sometimes necessary to use classical number theory in such proofs, still the converse approach is not adopted. Obviously this is now a formidable task, but results seem to be converging in this direction.

I think it would be worthwhile to consider some perspectives on arithmetike as the matrix for logistike, but now I shall make a preliminary investigation of some of the questions raised by the history of the extension of the number concept.

4: Changes in the concept of number

I shall now look at the way in which the concept of number was extended in the European rebirth of mathematics, some of the underlying attitudes and their implications. We have already briefly considered the Greek conceptualisation - their ontological distinction number:magnitude corresponding to the antinomy discrete:continuous. Before beginning to investigate some of the philosophical questions raised in connection with the changes in the concept of number, I shall first give a very summary history of the developments after the Greek era.

The Romans were primarily interested in practical results rather than theoretical mathematics and the continuing usage of their number system in the 'dark' ages (making multiplication and division extremely lengthy tasks) meant that there was little theoretical mathematical work in this period. Its rebirth was an important element in the phenomemon of the Renaissance. Theory was stimulated by the introduction of Arabic texts and Greek texts via Arabic translations. The gradual adoption of the Hindu-Arabic number system which contained a sign for zero and was a consistent place system, facilitated numerical operations.

It was in the sphere of commerce that this number system was first introduced: the main concern was correct, convenient operation with numbers, not a theoretical foundation. Since fractions arose in simple numerical calculations, they were popularly considered

numbers, i.e. the Greek discrete:continuous
number:magnitude distinction was bypassed. As
a result of the study of the Arabic texts
there was much interest in algebra, primarily
in solving polynomial equations; the terms
'cossike', 'surd' or 'absurd' numbers were
used variously for rationals, irrationals and
what we would now call the variable terms of
an equation - there was an implicit
assumption that polynomial equations were
determinate, that there was a definite
numerical solution waiting to be discovered
(or, as we would describe it, that the
existing number field was closed under the
operations used).

The use of the term 'number' for these cases
was disputed, in the first place by the
neoplatonists. The emergence of negative and
imaginary solutions caused further confusion.
There was in fact more difficulty in
accepting negatives than irrationals[1]; the
negatives lacked the framework which the
Eudoxan theory of proportions supplied for
the irrationals. From the general questioning
as to the criteria for acceptance of
candidates for numberhood, the dominant
perspective which emerged seems to have been
a pragmatic one: operations with the new
number-like entities continued because they
were useful. Stevin's position was one of the
most coherent; amongst others he championed
the decimal notation for fractions and
consequently advocated the radical notion
that 'Number is not all discontinuous
quantity'. This was the vague beginning of
the idea of a number line, a different kind
of parallel between geometry and arithmetic

from that of the Greeks. The existence of a
symbol for zero was of vital importance in
this; Stevin and later Wallis argued its
equivalence to the geometric point, as
opposed to the Greek equivalence of the
'monas' to the point. They were also both
concerned to be explicit about the
corresponding new status of 'one' - that it
should be considered a number, since,
according to Wallis, it answers the question,
'How many?'.

In the 17th century the question of the
conceptualisation of number was generally
secondary to problematics of the operational
developments. As a result of the more
pragmatic attitude (revealing the beginnings
of a formalist attitude to mathematics)
negatives and imaginaries were used like
other numbers in calculations because they
ultimately rendered correct results, even
though when they emerged themselves as
solutions, these were regarded as
meaningless.

These operations and the corresponding
attitude were not validated until the 19th
century, when Hamilton elaborated the
consistent algebra of complex numbers, and
negatives and imaginaries were accorded a
more 'intuitive', visual meaning in the
Gauss-Wessel representation of the complex
plane.

In the 18th century there had already been
attempts to prove the fundamental theorem of
algebra which implies that no new types of
candidate for numberhood could emerge from

polynomial equations; but Euler then (in an interesting reversal of the usual order of discovery preceding acceptance) postulated the existence of those transcendental numbers, i.e. that 'transcend the power of algebraic methods'. It was in the 19th century that first Liouville (in 1844) demonstrated that a certain serial form would yield transcendental numbers, and later Hermite (in 1873) showed the transcendence of e, and Lindemann (in 1882) that of π .

Both irrationals and transcendentals slip through the dense mesh of decimal fractional notation for the number line and at the end of the 19th century, Cantor developed a theory of transfinite numbers (cardinal numbers of infinities) which included a proof that the order of infinity of the continuum was higher than that of the whole numbers (and rationals). The development of the theory of these new numbers opened up controversies that harked back to the Eleatic paradoxes. Also, interestingly , a new monadic structure emerges in that the transfinite numbers are discrete: no-one has yet managed to construct a set whose cardinal number lies between that of the natural numbers and that of the real numbers, and higher transfinite numbers are generated by considering the set of subsets of a transfinite set.

Now to return to some of the philosophical questions raised. First, before considering any of the more particular questions, when examining Greek mathematics we can not ignore the context of the original mathematical

concern, which takes us into contradictions whose interplay has been a vital element in the growth of the mathematical organism.

For the Pythagorean-Platonic perspective (from which, rather than from the other Greek schools of though, our mathematics mostly derives) mathematics was intended primarily as a metaphysical discipline; the aim was to understand material reality as a manifestation of divine truth, not to control nature. The guiding principle was the order ($\tau \acute{a} \xi \iota s$) of the whole; the foundation of their metaphysics was a belief in the eternal unity, the 'one', and they sought through an understanding of changeless mathematical relations to attain to the highest truths pertaining to the eternal reality which transcends material reality.[2]

For the Pythagoreans, the transcendental, absolute truth was immanent in material, phenomenal reality; for Plato it lay behind or above worldly reality - there is a separation, an abstraction. For Pythagoras mathematics contained the truth; for Plato (although there is some ambiguity about his position) it seems that mathematics was a step towards an appreciation of the truth. For both, perception of the truth necessitated a self-transformation: it is not nature that hides, it is our vision that is skew.[3]

Since the time of the Greeks, mathematics has been identified with truth, but the nature of the truth sought has changed radically from being a revelation of transcendent reality,

to an undeniable, empirical statement, 'fact'[4] and even a system of tautologies; yet paradoxically, the one constant element in the idea of truth has been the quality of being absolute, timeless, unchanging! The underlying isues here touch the shared boundary, the no-man's-land between epistemology and ontology.

The Pythagorean-Platonic orientation was primarily ontological. The Platonic dialectic was a process to allow an ascent to a vision of the form of the good, a transcendent, subjective, absolute certainty, which was essentially incommunicable to others who did not take part in the ascent. Proclus' neoplatonist exposition of an 'ascent from more partial to more universal understandings' by which 'we climb up to the very science of "being" in so far as it is "being"'[5] was extremely influential in the Renaissance; but the telos behind this ideal of a science transcending other sciences was profoundly different from the original Platonic goal. Plato's vision of a unified metaphysical science was split. The subjective, metaphysical orientation was preserved in the opera of the alchemists; but these, for various reasons including the danger of persecution, largely remained privatised and that mode ultimately disappeared. Recently such works are being re-evaluated in terms of their effects on the new, emergent science - the rival mode which triumphed.

It is, of course, the Cartesian philosophy which most clearly indicates the reversals of

the classical Greek approach - as well as the
continuities. Plato's dialectic works through
accepted statements to attain a higher level
of absolute, subjective certainty; with
Descartes' epistemological orientation,
hereverses this order to found his
rationalism on subjective certainty of a low
level, then, taking mathematical reasoning,
the 'geometric' method as his model, he
believes it possible to proceed via clear-cut
self-evidences, to accumulate a higher level
of knowledge that is explicit, articulatable
and still absolutely certain.[6]

As Husserl[7] points out, there is a
fundamental inconsistency between Descartes'
radical starting-point, the epoche, and the
rationalist system that he develops (the
former destined ultimately to undermine the
latter); his bracketing is incomplete -
belief in the Galilean, mathematical book of
nature is not submitted. Before following any
further the theme of the nature of
mathematical truth and certainty, attempting
to unravel the cross-threadings of the
empirical and transcendental,
objective:subjective, relative:absolute etc.,
I shall return to consider the development of
the number concept and its relation to the
more general mathematical conceptualisation.

For the Pythagorean and Platonic
perspectives, the question of the ontological
status of number and mathematical concepts is
of prime importance. In the original
Pythagorean conceptualisation this status is
clear: all is number, where number is
discrete, heterogeneous, substance and form,

quality and quantity. The existence of the irrationals, which made this early vision of a mathematical universe untenable, was encompassed in Plato's hierarchy of levels of reality from the illusory, changeable, material world to the real, metaphysical, changeless realm of ideal forms, where the form of the good presides. The discrete:continuous, number:magnitude contradiction could be contained, if not clarified, in an ontological distinction. The beginnings of the modern conception of number can be seen in Aristotle's argument against the Platonic idealist conception, where he maintains that the unit is merely the measure of number. He also questions the equivalence of the point and the monas, but not the number:magnitude distinction. In this move away from a metaphysical, ontological foundation we see the beginning of the problematic of the status of number which became manifest in the Renaissance, eventually leading to a re-opening of the question as to the status of mathematical concepts generally.

When the question of the conceptualisation of number recurs in the Renaissance, we are at an extremely interesting historical juncture, since the problematic at this time could be seen as being both caused and resolved by pragmatism. As stated, fractions, as well as 'one' and zero had come to be regarded as numbers in everyday commercial usage, but this state of affairs was not so different from the everyday context of the Greek mathematical philosophers. The difference was that the Renaissance philosophies offered no

perspective on questions of the conceptualisation of number, which only became problematic in connection with the algebraic developments.

Following Diophantus' usage of the term 'arithmos' to represent the unknown in his investigations, where he also accepted fractional parts of the monas, it was assumed that the polynomial equations would yield numerical solutions. The strangeness of the emergent solutions, which were irreducible either to elements of the accepted number domain or to irrationals which corresponded to the Greek criterion of constructibility, caused a questioning of the earlier unselfconscious pragmatism and a concern to proscribe the limits of number: the right of fractions to numberhood was not questioned.

In retrospect, we might say that the acceptance of fractions as numbers necessitated the eventual acceptance of the other candidates for numberhood: but that is to assume the concept of a number field, a number system which is closed under certain operations. This criterion, which is linked with a formalist understanding of mathematics, emerged very slowly. It was not until the 19th century that it came to be articulated more explicitly and receive more general acceptance, although it began to bemanifest in the consciously pragmatic attitudes adopted in the face of the new offspring of number. Until that time widely divergent views continued to be voiced as to the status of the negatives, imaginaries and irrationals; such views decreased in

importance as the operational importance of the unplaced entities increased.

In the Renaissance climate of economic expansion (as opposed to the relatively static Greek economy) with its need for improved arithmetical techniques, and the belief in an effective, technologically oriented science based on the use of quantitative mathematical models, number was generally seen as quantity; the Greek ontological prohibition of fractions was no longer relevant, indeed it was scarcely considered; fractions appeared 'natural' (the Greek view of them as ratios survived in the nomenclature 'rational' numbers); negatives were considered variously 'absurd', 'impossible', 'fictitious' and 'false', whilst the term 'imaginary' was coined dismissively by Descartes, complex numbers having been described earlier as 'useless' and 'sophistic'.[8]

All these terms are revealing as regards the implied grounds for accepting candidates for numberhood; on the one hand there is an assumption of a sane, possible, real, true area prescribed by the accepted, positive, rational numbers (whose ontology is, at that time, not questioned); then, the term 'useless' betrays an incipient teleology in mathematics, it signals the beginnings of a more conscious pragmatism.

In the first place it is a contradictory pragmatism: Cardan denigrates imaginary numbers as 'useless' and continues to use them ('putting aside the mental tortures

involved')[9];   theoretical attitudes oppose
operational practice. The practice continues
and in Girard's position we see the
theoretical attitude reversed to form a
coherent pragmatism: he asks himself[10]:

> Of what use are these impossible solutions
> [complex roots]? I answer: For three
> things   - for the certitude of the
> general rules, for their utility, and
> because there are not other solutions.

He frames his question pragmatically and
answers himself partially tautologically; his
futher justification reveals a rudimentary
formalism. From the perspective of modern
rigour, this claim of certainty, and the
assumption that the extant solutions complete
the system, is unwarranted. The claim of
certainty accompanied the use of the new
numerical offspring throughout the long
period of operation with them while their
status was not agreed, not determined. It is
only with the rebirth of rigour that the
notion of certainty begins to become more
specific. In the formalist doctrine, it
finally returns to revive its original Greek
counterpart: non-contradiction again becomes
the paradigm, but the locus of fundamental
validation is reduced; for the Greeks it was
a consistent, global ontology; in the modern
age completeness is required not in a global
philosophy, rather it is sought in the local
mathematical microcosm[11] - the original
formalist demand being that a mathematical
system be consistent and complete. Girard's
theoretical attitude articulated the practice
of the time; most mathematicians continued to

practise pragmatism in contradiction with their theoretical positions.[12]

The status of the irrationals was also re-evaluated. Some mathematicians (including Pascal and Barrow) wished to salvage the number:magnitude distinction although its original foundations were no longer valid ontologically or technically.[13] But there was also an attempt to articulate a new criterion for the acceptability of candidates for numberhood, more in keeping with the quantifying spirit of the Renaissance, based on the extension of decimal notation to include fractions: Stifel first gives the pragmatic reasons that might be given for accepting irrationals as numbers:

> Since, in proving geometric figures, when rational numbers fail us, irrational numbers take their place and prove exactly those things which rational numbers could not prove.... we are moved and compelled to assert that they truly are numbers, compelled that is, by the results which follow from their use - results which we perceive to be real, certain and constant.

But then he argues that:

> Other considerations compel us to deny that irrational numbers are numbers at all. To wit, when we seek to subject them to numeration [decimal representation] .... we find that they flee away perpetually, so that not one of them can be apprehended precisely in itself ... Now that can not be called a true number

which is of such a nature that it lacks
precision... Therefore, just as an
infinite number is not a number, so an
irrational number is not a true number,
but lies in a kind of a cloud of
infinity.[14]

His subsequent argument that 'true' numbers
are either whole numbers or fractions is
circular, in that the decimal fractional
notation is specifically constucted so as to
express fractions in terms of ordered
sequences of integers, and, in fact, contains
a technical inconsistency in that he omits to
point out the distinction between the
ultimately predictable iterative procedure by
which fractions whose denominators are prime
to ten, when expressed in decimal notation,
similarly recede to infinity, and the
unpredictable course of the irrationals in
such a representation. But in his attempt to
abstract a criterion for acceptance of
candidates to numberhood, that of precise
locatability in terms of the extended decimal
notation, he reveals an awareness of the
problematic of the limit concept which is
ignored in Stevin's later conception.

Ultimately Stifel's locatability criterion,
like Descartes' argument for the limited
acceptance of negative numbers (that
equations with 'false', i.e. negative, roots
can be transformed so as to yield positive
roots), does not come to terms with the issue
of the conceptualisation of number, but
rather refers back to a presupposed, accepted
number domain. In fact, Descartes does deal
more directly with some important aspects of

the number concept, but before discussing
these we must first review Stevin's position.

Stevin's conception of his mathematical task
as part of a general project to recover the
knowledge of a 'wise age' that he believed to
have existed before the Greeks[15] (whose
culture he sees as the beginning of a
'barbarous age') compelled him to explicate
thoroughly his radical conceptualisation of
number, with reference to the surviving
formulations of the Greeks.

Stevin was not the first to use the decimal
fractional notation (Vieta and Bombelli
amongst others used versions of such a
notation) but his **Disme**, which functioned as a
teaching manual for calculating with decimals
and as propaganda advocating decimal
standardisations of measures, was extremely
influential[16] in spreading the use of the
notation which clearly plays a very important
role in determining his conception of number.

His first definition of number (articulating
the spirit of the time) is 'that by which the
quantity of each thing is expressed'.[17] He
then posits an analogy between number and
continuous, homogeneous matter, stating his
fundamental premise that the part is 'of the
same material'('de mesme matiere') as the
whole.[18] Then, using the classical definition
of number as a multitude of units, he argues
that the unit, being a part of a multitude of
units, is of the same material as the
multitude of units, but the material of the
multitude of units is 'number', so the
material of the unit, and thus the unit

itself, is 'number' - not the principle or beginning of number (as it is in the classical Greek view). He remarks by way of illustration that to deny this last step would be like denying that a piece of bread is bread; the devaluation of the integers (which he later makes explicit) is already clear - this conception of number does not allow for the qualitative difference between a loaf of bread and a pile of crumbs!

For Stevin, the unit is divisible (he invokes Diophantus); zero is now the beginning of number and the analogue of the geometric point. He states explicitly that 'number is by no means discontinuous'; number and magnitude are now so similar as to be almost identical; he attacks the use of the terms 'absurd', 'irrational' for incommensurables - any root of a number is a number, since it is a part of a number. He retains a disinction between 'arithmetic' number, 'one expressed without an adjective of size' and 'geometric' numbers, 'quadratic', 'cubic' numbers etc.: any 'arithmetic' number may be a 'geometric' number, but when the numerical value is not known, 'geometric' numbers represent the indeterminate quantities in algebraic calculations, and are denoted by ①$_2$, ②$_{3'}$ ③ etc. (where we would write x, x^2, x^3 etc.); he thinks not in terms of 'variables', but of 'unknowns' which are still geometrically cloaked.

Stevin, with his practical background, was primarily interested in determinate solutions to problems; Vieta's approach was different. Like Stevin, he saw his work as a recovery of

lost knowledge not as creation[19], he saw in the study of polynomial equations the possibility of a general mathematical method. In his **ars analytice** (the explicit telos of which was 'to leave no problem unsolved') he extended the Diophantine algebra (which he saw as revealing the methods which Diophantus unconciously used but concealed), extracting general methods from Diophantus' particular cases. In the symbolic notation which he began to develop for polynomial equations (although in the **ars analytice** he does not use a sign for equality; he verbalises the progressive steps; it is not our modern equation form), the unknown is clearly distinguished by letter. He retains the terms 'side', 'square', 'cube', 'square-squared' verbally and refers to the quantities with which he deals, as 'magnitudes'.

We are now in a position to return to Descartes (in whom we again find the belief in an earlier, more complete knowledge[20]). Although the extent to which Descartes was influenced by Stevin and Vieta is not clear, his work could be seen, both on a mathematical, operational level, and on a more general philosophical level, as combining and extending theirs.

As regards mathematics 'proper', it is Descartes who liberates algebra from its internal geometricisation (which had become increasingly a relic from the Greek formulation) both on an elemental, notational level - our modern algebraic notation derives from his synthesis of Vieta's literal notation for the indeterminate term and

Stevin's use of numbers to denote the power;
and on an operational, conceptual level -
considering the quantity as distinct from the
geometric status (as Stevin had done for
determinate number, but not in the case of
the variable) - he explicitly states that a
product of lines can be a line.[21] Then,
having purged algebra of its geometric
residue, he is able to establish a new
relationship, at a higher structural level,
between geometry and the algebra of
dimensionless measure, a correspondence
between equations and geometric curves; he
forms a new synthesis, a new mode, coordinate
geometry.

Descartes himself saw this as an example of
the practical effectiveness of his general
method which was aimed at the 'mathesis
universalis', a general science of order and
measurement, that could be seen as a further
stage in achieving the generality which Vieta
envisaged in his **ars analytice**. Descartes
founds his mathesis on a substantiation of
Stevin's number-matter analogy; it is no
longer seen as a metaphor; on the basis of
his psycho-physiological model Descartes
argues that there is an exact, real
correspondence between number and matter;
extension is both symbolic, as the object of
general algebra, and real, as the substance
of the corporeal world.[22]

Descartes thus supplies a philosophical and
technical foundation for the budding science,
in his articulation of the rationalist method
and his identification of the basic subject
matter of abstract mathematics with that of

science, the investigation of the material world, and in his structural quantification of Euclidean space which prepares the ground for the differential calculus, a mathematical technique for dealing with mechanical, quantitative change. A space ordered by a continuous (potentially infinitely divisible) measure was a necessary prerequisite for this development, as was a homogeneous time - a vitally important factor in the growth of Renaissance science. The homogeneity imposed upon space was imposed upon time.[23] Without the denial of the essential difference between these two basic orders of the life-world, the mechanical mathematics of the differential calculus is unthinkable. This is the crux of the Eleatic paradoxes. The Greek mathematical solution was to homogenise space and ignore time, deny change. The mathematical mode engendered could entertain an integral calculus, the method of exhaustion for accumulative approximation to the space contained by a curve (line or surface), but not investigation of a point phenomenon.

In the Renaissance, time enters as a concious, explicit concern of science: the static, Greek episteme gives way to Renaissance, time-, effect-oriented science. In mathematics the antinomy determinate:indeterminate overlays the discrete:continuous contradiction which dominated Greek mathematics. The Greek conception of number was architecturally spatial; number was composed of geometrical arrangements of monads, indivisible units. The elements of arithmetike embodied spatial

forms. Time is introduced into the conceptual
material of mathematics (and number) through
polynomial equations (as distinct from its
explicit, external entry as an object of
study). Originally they were seen as
determinate; their subject was the 'unknown'
which was specified geometrically. The change
in perspective by which the 'unknown' became
the 'variable' simultaneously recognised time
and stepped outside of it; the determinate
solution became subordinate to the vision of
the form of the possibilities of solution.

This change in perspective is already
inherent in the new notion of number as
articulated by Stevin; number is homogeneous
material, a conceptual object and the
material which constitutes the object. The
discrete:continuous contradiction appears to
be effaced. Stevin in fact argues the
relativity of incommensurability on the
grounds that a length that is commensurable
in one place, could be incommensurable in
another with a different standard unit; this
fact, which is due to the relative nature of
dimensional units, does not eradicate the
contradiction inherent in number. Stevin's
formulation articulates the intuitive notion
of his time; his concept of number is closer
to the Greek magnitude, continuous measure
(viz. Vieta's use of the term 'magnitude' for
the subject of a polynomial equation); the
absoluteness of the unit, the Greek
foundation for discrete, heterogeneous number
is undermined. The original integral decimal
notation emphasised the homogeneous aspect of
the integers, the repetitive procedure for
approaching the potential infinity of

succession; the extension of the notation in
the other direction gives an intuitive sense
to the infinity of density, again emphasising
the homogeneity. The decimal fractional
notation, with its indefinitely close
approximations to incommensurables encourages
the illusion of a smooth elision from
discrete to continuous, with the concomitant
devaluation of the integers. Descartes'
explication of the latent idea of number-line
completes the image. Decimals are originally
constructed from integers; once they are
constructed, the integers appear to be made
up of decimals as (evanescent) building
blocks or (a more coherent view permitted by
Descartes' model) to be arbitrary places on a
number line. In fact, at each decimal place
one meets only another level of units; the
place of transition from discrete to
continuous continually recedes; one never
confronts the essential difference between
exact numbers and incommensurables; it is
merely postponed indefinitely. Smooth number
glosses over the integer:magnitude
distinction; the Greek hypostatisation which
had contained the discrete:continuous duality
by keeping the two poles rigidly apart is now
dissolved. Its continuing relevance to
considerations of the integers is obscured.
It appears temporarily to have been banished;
but it is merely transferred, transformed
into an internal contradiction of the
expanded number concept, which makes it
possible for the calculus to approach the
limit point - under probings the
contradiction again explodes.

With the suppression of the discrete, the
contradiction undergoes a modal shift: the
place formerly occupied by the discrete - the
number object - now houses determinate,
finite number, any specific representative of
number the material, the continuous, which
represents the potentially infinite
divisibility and/or range possible for any
determinate number; it is simultaneously the
material from which any determinate number is
formed and the homogeneous number line (or
space) from which a determinate number may be
chosen. Stevin states that every 'arithmetic'
(determinate) number is the beginning of
'geometric' (indeterminate) number, just as
zero is the beginning of 'arithmetic'
number.[24] Whereas the Greek number:magnitude
distinction was a static, horizontal dualism
fixed in space, the new concept of number
implicitly contains the notion of a variable.
The contradiction operates between the
levels, determinate and indeterminate; it is
vertical rather than horizontal, dialectical
not static. The new model fuses space and
time in common homogeneity: determinate
number is both formed of, and chosen from
homogeneous, potential number; the Greek
spatial composition of number persists in the
identification of a determinate number with a
line segment; but a determinate number is
also a place, a point on a number line which
may be singled out like a moment of time from
which the probable past and the possible
future stretch endlessly away. The
implications of this mode of fusion are
immense and warrant further discussion, but
immediately we see the root of the
problematic that concerned Miller and

Meyerson; the new mathematics initiated in the Renaissance can deal with time by reducing it to quantified space; the past and future are arbitrary in this schema (unlike the life-world) - dependent on an arbitrary origin; the illusion is created of the possibility of stepping outside time (an illusion which, like the fiction of Euclidean space itself, is valid only within certain limits); a framework is created for the quantitative description (and thus possible prediction) of mechanical change, but the essence of experienced time, real change, the emergence of the qualitatively new, still eludes description. This problematic has recently been approached within mathematics by René Thom; his catastrophe theory can provide qualitative models for changes of state within a certain nexus, but he himself considers that the theory is essentially incapable of adaption to quantification for predictive scientific purposes. The Renaissance vision of knowledge that is both certain and effective is reaching the limits of the mode it engendered.

It would be worthwhile to return to examine more closely the roots of present mathematical problematics in the Renaissance incunabular of our mathematical mode, as well as the continuities and reversals from the original Greek seeds. There are, in fact several interesting parallels between the Greek beginnings and the Renaissance rebirth.

The Pythagorean mathematical inspiration was the vision of a numerical description of the world, where 'number' was discrete,

heterogeneous and in some way material. This
thesis met with contradiction in the form of
the irrationals;   the idea of number as
discrete was not sacrificed - that would have
meant abdication of metaphysics as a change-
less reality; but a new rigour was necessary
to ensure non-contradiction (equated with
certainty); on the one hand it was necessary
to supply a foundation for operation with
irrationals; on the other, the general form
of mathematical demonstration was tightened.
In the Renaissance twist of the spiral, there
is still a vision of a numerical world-
description - 'number' is now abstract and
continuous,   having   subsumed   the
discrete:continuous antinomy - but the new
impetus derives from the perception   of
mathematics as an epistemological method, a
general art for solving problems.

The Pythagorean vision is of an isomorphism
between the realm of discrete number and the
life-world;   the demonstrative method   is
secondary. Descartes reiterates this vision,
with   continuous   number   replacing   the
discrete, interposing a level of abstraction;
he reverses priorities - his epistemological,
rationalist   method   modelled   on   the
mathematical proof form is primary; he also
reverses the role of the form - in its
original context the theorem precedes the
proof, its direct function is static (it is
only through a further act of reflection on
the internal structural components of the
proof, an involution, in accordance with
Lakatos' proof analysis, that it receives a
function in generating new understandings);
when Descartes appropriates the form   he

transforms it into an epistemological method
whose function is generative.[25] In these
reversals, the monadic structure retains its
primacy: for the Pythagorean, it is the
ontological monadology, the unit elements,
the prime constituents of the phenomenal
world; for Descartes, it is epistemological -
the constituents are the clear and distinct
ideas which accumulate to form a body of sure
knowledge.

There is a further parallel, both with regard
to content and history, between the
(Pythagorean) Platonic metaphysics and the
Cartesian rationalist science. Both deny
real, qualitative change, the first by
denying time, the second, by neutering it;
both proved successful within limits; their
very success caused the ideas to become
embedded as ideology, so that even when the
limits of their validity are approached and
the original doctrines are questioned, their
consequences still survive in mathematical
praxis of which the roots have been
forgotten, sedimented in history.

The rebirth of Greek mathematics in Western
culture is simultaneously a completion and an
inversion. The development of the decimal
notation fosters the new concept of
homogeneous number, allowing the subsumption
of time into mathematics. The initial
consistent place system, of course, depends
on the existence of a symbol for zero, and it
would be worthwhile considering the further
implications of this, for instance, the
change of meaning that occurred in the
cultural transfer of zero from its Hindu

origins, where there is a sense of a full
nothingness, to a society where 'nothing' is
a mere absence - where 'nature abhors a
vacuum' etc. What are the implications of our
conception of zero and those of the
alternative? Such questions obviously relate
to considerations of the calculus.
Interestingly Buddhist logic, beginning from
almost the opposite metaphysics to the
Platonist, with the point-instant as the
basic reality, came very close to notions of
a differential calculus.

The mode of the symbolic notation developed
in the Renaissance could obviously be
examined in more detail. For instance, before
the Renaissance, the equation was not the
paradigm of mathematics. The primacy of the
equation (which has only very recently come
to be questioned) internalises the mode of
the axiomatic method - static consolidation
of atemporal positivities.[26] The question of
overdetermination of mathematical symbols
needs to be investigated. When we begin to
look at mathematics as a language, we see
that the concept of number which emerges in
the Renaissance is adjectival with respect to
ordinary language. The number language then
bears a skew relation to ordinary language,
since these adjectives are then accorded a
substantive function in mathematical grammar.
The situation is, of course, vastly more
complex - mathematical grammar is not
isomorphic to the grammar of ordinary
language, but the attempt to understand
mathematics in this way is valuable - the
divergence between mathematical language and
ordinary language only began with the

development of symbolic notation in the
Renaissance, before that time mathematics was
still predominantly verbal. In classical
Greece mathematical language was embedded in
ordinary language; to understand mathematical
objects was to locate them in a global
ontology; numbers attained reality as
objects, and so could coherently be
understood as substantives, by embodying
geometric forms. In the Renaissance
mathematical operations are symbolised;
mathematical language is thus formally
separated from verbal language. It has an
internal coherence, being now composed
homogeneously of symbolic elements; it is
free to follow its own dynamic according to
its own, internal, grammatical and
syntactical laws. In the new reflexivity of
mathematics the number concept is extended by
internalising mathematical operations; an
operational classification of number
supersedes the Greek geometric
classification.

In the initial phases of growth of this mode,
as Klein says:

> the whole complex of ontological problems
> which surrounds the ancient concept of
> number loses its object in the context of
> the symbolic conception, since there is no
> immediate occasion for questioning the
> mode of being of the 'symbol' itself.[27]

The extent of the rupture is not, at first,
recognised; Descartes' identification of the
mathematical object-world with the perceived
material world and Kant's attempt to refound

it metaphysically, is not called into
question until the development of consistent
non-Euclidean geometries, causing attempts to
justify what has by that time become the
status quo - conventionalism, logicism,
formalism. These explicitly refuse to
consider the problematic of the relation
between mathematics and the life-world,
attempting instead to create a Frankenstein's
monster of mathematics, a self-sufficient
entity whose judgment is more certain than
that of its creators. We must return to
examine more closely the ways in which the
various attempts failed to achieve their
original stated aims, but the very fact of
the failures must cause a re-examination of
the basis for such attempts. When
Wittgenstein questions the motivation behind
the foundations fervour, he asks:

> But what was the attempt made for? - Was
> it not due to an uncertainty in another
> place?[28]

Is that 'other place' not the questionable
place of mathematics itself in the life-
world, entailing consideration of such
problematic concepts as intuition and
certainty which bridge the objective and
subjective? The problematic posed for the
Greeks by the symbolic status of number, as
to its ontological reality, was temporarily
submerged as the whole of mathematics assumed
a symbolic character. Thus, the problematic
now recurs on a larger scale, of the
ontological status of mathematics.

Notes to Section 1

1. Since I wrote this, Philip Kitcher (**1983**) has begun to approach this area.

2. This has been abundantly demonstrated by Becker (**1927**), Lakatos (**1963/4**), Fisher (**1966/7**), Dessanti (**1968**), Foucault (**1970**) and Grabiner (**1974**) amongst others.

3. See Gödel (**1944**).

4. This is a phenomenological term, cf. Husserl (**1970**).

5. See, for example, Plato's **Republic** (Part 8, Book 7) and Farrington (**1946**) for his reference to the 8th book of Plutarch's Dinner-Table Discussions where Lycurgus is mentioned as allowing geometry but not arithmetic as a study in Sparta, because of the preferable political implications of the former.

6. See Husserl (**1946**) and Klein (**1940**). Also Barfield (**1979**) in his essay 'The Force of Habit' (particularly pp.69-79), vividly describes the imprisonment we experience as a result of our culture-based mental habits. He also (like

Pythagoras, Plato and Descartes amongst others) emphasises that we need to go through a certain self-transformation, both to realise the extent of our imprisonment and to begin to release ourselves.

7. See especially Fisher (1966/7).

Notes to Section 2

1. Bourbaki (**1969**) is one of the clearest
    examples of this Whig approach to the
    history of mathematics. See Unguru (**1975**)
    for a description of the results of such
    a perspective on the history of Greek
    mathematics.

2. See Kuhn (**1962**), particularly ch.11.

3. Wilder (**1974**) attempts to come to terms
    with this issue. See also Unguru (**1975**).

4. D.L.Miller (**1948**).

5. See Husserl (**1970**) and Barfield (**1957**).
    I think the basic fallacy underlying this
    (usually unconscious) assumption, is
    pinpointed by a Sufi saying which Keith
    Critchlow quoted in a Wrekin Trust talk
    in June 1986, namely, 'Conclusive is not
    necessarily exclusive'.

    The fact that one explanation of a
    phenomenon works well does not imply that
    it is the only possible explanation (that
    it is true and all others are false). A
    simple example (which is nonetheless
    valid) is that of the picture (often seen
    in textbooks on the psychology of

perception) of two black head profiles
facing each other, separated by a white
vase outline. It is not a case of
either/or; both views are valid.

6. See Koyré (1957).

7. See Wittgenstein (1956), Cameron (1970)
and Bremer (1973).

8. Klein (1940).

9. Husserl (1970). See also the discussion
of nomological and eidetic disciplines in
Husserl (1913).

10. Husserl (1929).

11. See Lakatos (1963/4) and Polya (1954).
Lakatos demonstrates this process of
increasing inclusion. See also Whiteside
(1960/2).Einstein with respect to Newton
is another clear example. It is
interesting and unfortunate that although
the more inclusive concepts or theories
do not contradict earlier more partial
ones, yet they are usually seen at first
as threatening.

12. Szabó (1969).

13. See Szabó (1969). See also Whiteside (1960/2) for a discussion and illustrations of this shift as it affected mathematical proofs from the 17th century into the 18th. The question of the validity of proof and the meaning of proof obviously recurs now with extra bite with the computer proof of the 4-colour theorem and the collaboration proof of the classification theorem for finite groups.

I shall take this opportunity to discuss an area which I could, perhaps, have brought into the main text. When I read Szabó (1958) on the shift from illustrative proofs to more strictly logical ones, it made me wonder when the term 'demonstration' was replaced by 'proof'. (I have not yet found out and I would be grateful to anyone who does know if they would get in touch with me to let me know.) The term 'demonstration' implies the attempt to convince someone; the term 'proof' implies certainty on the part of the prover. The term 'proof' assumes that there is an objective truth or objective rationality independent of any subjective consciouness assessing the rational process.

14. Szabó (**1969**).

15. For example, the technical consequences
    of some of Kronecker's arguments (e.g. on
    Aristotelian logic and on the centrality
    to mathematics of the natural numbers)
    were fully realised only in the work of
    Brouwer and the intuitionists.

    It is ironic that Aristotle was clear
    about the limits of validity of his
    logic; it is those who have used it
    subsequently who have implicitly assumed
    it to have global validity.

Notes to Section 3

1. This is certainly what the Löwenheim-Skolem theorem says.

2. Or even work in the philosophy of mathematics. One exception is Becker (**1927**).

3. This attitude which has been shared, in one form or another, by all the great figures in modern philosophy of mathematics from Frege (for whom 'Wert-verlaufe' take the place of sets) onwards, has been criticised by Wang who has pointed out that no substantial reasons have been given for supposing that 'numbers evaporate while sets are rocks' (Wang, **1974**, p.238).

   See also the more general criticisms of 'foundationalism' in Smith (**1976**) and the references given there.

4. Wittgenstein (**1956**) p.106 (II-82).

5. The distinction made by Klein (**1968**) between number theory and arithmetike has some validity and leads to certain interesting questions but they are not immediately relevant to the present discussion.

6. For example, as Aristotle tells us, 'linear', 'plane' etc. (**Metaphysics,** Δ.14.1020a35-b8).

7. See Dobbs (**1975**) who shows that Newton's alchemical work preceded and was concurrent with his scientific research. It was previously assumed that he only became an alchemist later in his life when he had completed his scientifically acceptable work; this assumption made it possible simply to dismiss the alchemical work as the product of his dotage.

8. Aristotle, **Metaphysics,** A8, 990a 3f.

9. Aristotle, **Metaphysics,** M6, 1080b 19f.

10. The importance of unity, oneness, wholeness, is beginning to be recognised again with the growth of holistic approaches to knowledge. In biology, clearly, the whole is greater than the sum of its parts. I believe that we also need to begin to investigate 2-ness, 3-ness etc. in similar ways to those indicated by M.-L. von Franz (**1974**), using the wealth of modern mathematical knowledge, mostly logistike, to uncover the corresponding arithmetike.

11. These issues are discussed in detail by
    Becker (**1927**),pp.129-148 and pp.199-213.

12. Szabó (**1958**),pp.118-20.

13. See Klein (**1968**) for more detail.

Notes to Section 4

1. See Becker (**1927**) and Kline (**1972**).

2. See Becker (**1927**), Cornford (**1939**), Klein (**1968**). The shift in attitude is clear if we look at the original meaning of 'theory'; as Russell says (**1971**,p.52) 'this was originally an Orphic word, which Cornford (**From Religion to Philosophy**) interprets as "passionate sympathetic contemplation". In this state, he says, "The spectator is ident- ified with the suffering God, dies in his death, and rises again in his new birth." For Pythagoras, the "passionate sympathetic contemplation" was intellectual, and issued in mathematical knowledge.'

3. The term 'mathematics' was coined in Pythagoras' school, τὰ μαθήματα (ta mathemata) meaning 'those things which have been learned'. Given the religious nature of the Pythagorean school, it is certainly arguable that the fundamental learning concerned the spiritual development of the disciples and what we now consider to be the mathematics deriving from that school were symbols of that spiritual or psychological development.

It is interesting that the original
etymological meaning of mathematics is so
open. This may partially account for one
problem which has dogged mathematical
philosophy for some centuries (although
it has not always been acknowledged),
namely that mathematics has been defined
by extension not by intension (to return
to a neglected but useful Aristotelian
distinction).

4. See Lewis Mumford, (1970). It is
    interesting that the etymological roots
    of 'fact' and 'fiction' are remarkably
    similar; 'fact' derives from the Latin,
    'facere', to do, to make; 'fiction'
    derives from the Latin, 'fingere', to
    form, to fashion.

5. See Klein (1968).

6. In his Discourse on the Method of Right
    Reasoning, Descartes says (Descartes,
    1973, p.92),'Those long chains of
    reasoning, simple and easy as they are,
    of which geometricians make use in order
    to arrive at the most difficult
    demonstrations, had caused me to imagine
    that all those things which fall under
    the cognizance of man might very likely
    be mutually related in the same fashion;
    and that, provided only that we abstain
    from receiving anything as true which is

not so, and always retain the order which
is necessary in order to deduce the one
conclusion from the other, there can be
nothing so remote that we cannot reach to
it, nor so recondite that we can not
discover it.'

I am not a Cartesian scholar, but in my
limited reading of and about Descartes, I
have found no comments on what seems to
me to be most remarkable: the 'geometric'
method was the deductive method, i.e. a
method to prove a theorem once it exists,
Descartes writes as if it were inductive,
i.e.   a   method   of   discovering   or
generating new results.

Something I recently discovered from a
prominent Cartesian scholar was   that
Descartes intended his method to be used
only after his course of meditations had
been followed. This part of his teaching
appears to have been totally ignored by
his disciples in his lifetime and
subsequently.

7. Husserl (**1970**),pt.II, ch.18, p.79ff.

8. Kline (**1972**), p.253.

9. Ibid.

10. Ibid.

11. My stance may be seen as somewhat naive, but I consider that once this reduction in philosophical demand is recognised, it can not be viewed other than as an impoverishment.

12. Girard was an oddball in his time in his attempt to be consistent; since then the split between doing and being has continued, deepened.

13. See Whiteside (**1960-62**).

14. Kline (**1972**), p.251.

15. Klein (**1968**), pp.186-90.

16. The Disme was written in the vernacular. Stevin was one of the first to make this important change.

17. Klein (**1968**).

18. Klein (**1968**). It is interesting that both Stevin and Descartes assume matter to be homogeneous.

19. Klein (**1968**).

20. Klein (**1968**).

21. Descartes (**1968**) p.169.

22. Klein (**1968**) p.210-11.

23. See Meyerson (**1930**).

24. Klein (**1968**). I think that further investigation of this profound conceptual shift could yield greater understanding not only of the ontology of numbers and mathematics, but also of the background consciousness of our current mathscientific culture.

25. As stated in note 6 of section 4, this seems to have been more of a misunderstanding on Descartes' part than a deliberate step.

26. See Adorno and Horkheimer (**1944**), p.7 ff. for a description of the mutual mirroring of mathematical and societal developments e.g., 'Bourgeois society is ruled by equivalence. It makes the dissimilar comparable by reducing it to abstract quantities. To the Enlightenment, that which does not reduce to numbers, and ultimately to the one, becomes illusion; modern positivism writes it off as

literature. Unity is the slogan from
Parmenides to Russell. The destruction of
gods and qualities alike is insisted
upon.'

27. Klein (**1968**).

28. Wittgenstein (**1956**).

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